Describing fundamental groups of Polish spaces

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It is a theorem of [Shelah 1988] that for a path-connected, locally path-connected compact metric space X, $\pi_1(X)$ is either finitely generated or uncountable.

- Hatcher, *Algebraic Topology*

Shelah's Theorem

Suppose X is a path-connected and locally path-connected compact metric space (aka a Peano continuum).

(i) If X is *semi-locally simply connected*, then $\pi_1(X)$ is finitely generated.

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Shelah [8] originally proved this by forcing and absoluteness argument. Pawlikowski [7] gave an easier proof using classical DST.

Fundamental group

A space X is *path-connected* if for any $x, y \in X$, there exists $f : [0,1] \to X$ s.t. $f(0) = x$ and $f(1) = y$. It is *locally path-connected* if path-connected open sets form a basis.

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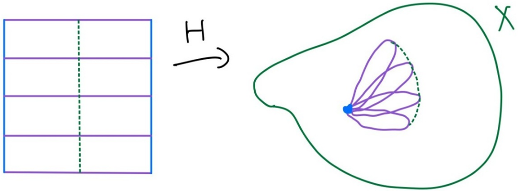
Fix a base point $x_0 \in X$. A *loop* in X is an $f : [0,1] \to X$ s.t. $f(0) = f(1) = x_0$. ΩX is the Polish space of all loops in X.

Fundamental group

$f, g \in \Omega X$ are *homotopic* if there exists $H : [0,1] \times [0,1] \to X$ s.t. $H(s,0) = f(s)$, $H(s,1) = g(s)$, and $H(0,t) = H(1,t) = x_0$ for all $t \in [0,1]$.



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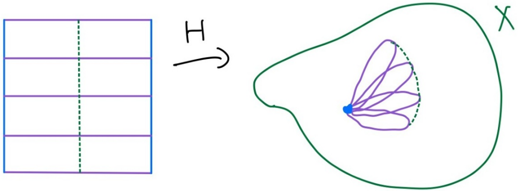
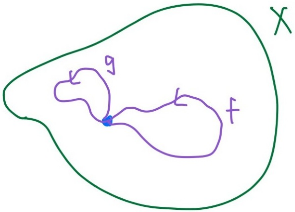


This is an analytic equivalence relation on ΩX, denoted $\sim_X$, and the equivalence class of f is denoted $[f]$. $\pi_1(X) = \Omega X / \sim_X$ is the collection of all equivalence classes.

Fundamental group

For $f, g \in \Omega X$ define $f \cdot g$ by:

$$f \cdot g(s) = \begin{cases} f(2s), 0 \leq s \leq 1/2 \\ g(2s-1), 1/2 \leq s \leq 1 \end{cases}$$



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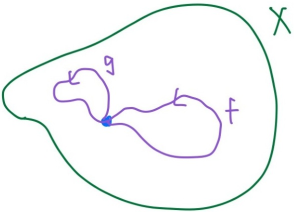

This induces an operation on $\pi_1(X)$ by $[f] \cdot [g] := [f \cdot g]$. The identity element is (the equivalence class of) the constant map $f(s) = x_0$, and the inverse of $[f]$ is represented by $\bar{f}(s) = f(1-s)$.

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A base-point preserving map $\Phi : X \to Y$ induces a group homomorphism $\Phi_* : \pi_1(X) \to \pi_1(Y),\ [f] \mapsto [\Phi \circ f]$.

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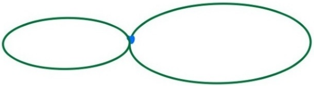

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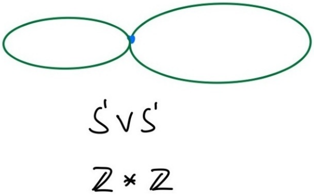

Similarly $\pi_1(\bigvee_{\alpha \in I} S_\alpha)$ is the free group on $|I|$ many generators.

Examples

Given $f : Y \to X$, the *mapping cone* is obtained from $Y \times [0,1] \sqcup X$ by identifying $(y,1)$ with $f(y)$ and collapsing $Y \times \{0\}$ to a point.

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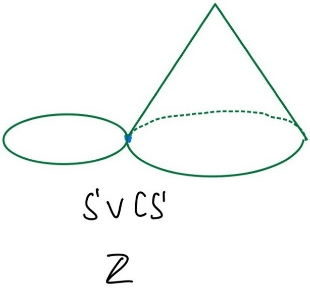
Attaching mapping cones to $\bigvee_{\alpha \in I} S_\alpha$ can create any desired fundamental group.



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X is *semi-locally simply connected* (slsc) if each point x_0 has a neighborhood U s.t. the map $i_* : \pi_1(U) \to \pi_1(X)$ induced by inclusion $i : U \to X$ is trivial, i.e., any loop at x_0 in U can be homotoped to the constant map in X.

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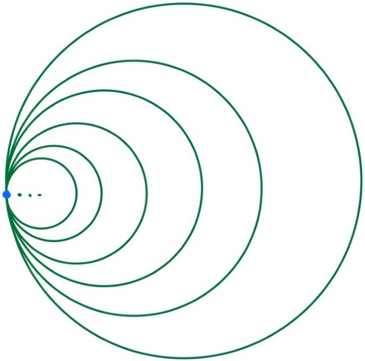
Examples of lsc spaces: manifolds, CW complexes

Connectedness

Non-slsc: the Earring space $\mathbb{E} = \bigcup_{n \geq 1} C_n$, where C_n is the circle $(x - 1/n)^2 + y^2 = (1/n)^2$.



$\pi_1(\mathbb{E})$ is quite wild. For example an $f : [0, 1] \to \mathbb{E}$ can traverse C_n during $[1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}]$.

Shelah's Theorem

Theorem (Shelah)

Suppose X is a path-connected and locally path-connected compact metric space.
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It has been shown that $\pi_1(X)$ is in fact finitely presented [2, 4].

Shelah's Theorem

Second part: not slsc means there is a point $x_0 \in X$ s.t. any neighborhood $U \ni x$ contains an *essential loop*, namely one that is non-trivial in X. So around that point X looks like the Earring space $\mathbb{E}$.

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Let f_n be an essential loop at x_0 with diameter $\leq 1/(n+1)$. For $c \in 2^\omega$, define $f_c : [0,1] \to X$ as follows: on $[1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}]$, if $c(n) = 0$ then f_c stays at x_0; if $c(n) = 1$ then f_c traces f_n.

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If c_1, c_2 differ at exactly one digit, then f_{c_1}, f_{c_2} are non-homotopic. The result then follows from Mycielski's Theorem and:

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Proposition

Suppose E is an analytic equivalence relation on 2^ω s.t. $\neg c_1 E c_2$ whenever they differ at exactly one digit. Then E is meager.

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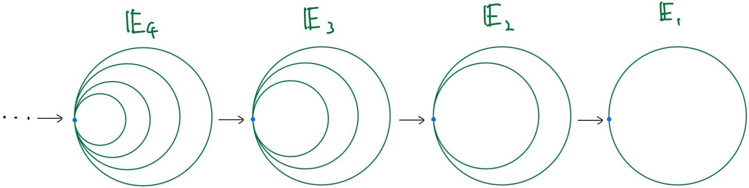

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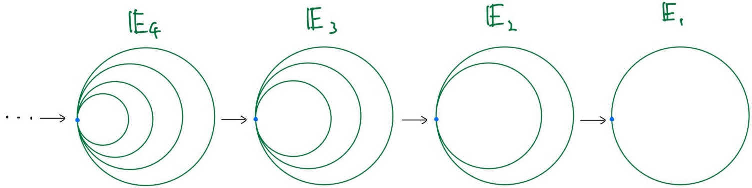

Let $\mathbb{E}_n = \bigcup_{1 \leq m \leq n} C_m$. $\pi_1(\mathbb{E}_n)$ is $F_n = \langle g_1, \ldots, g_n \rangle$, the free group on n generators.

The Earring group

$$\begin{array}{ccccccc} & & \mathbb{E} & & & & \\ & \swarrow & \downarrow & \searrow & & \searrow & \\ \cdots & \longrightarrow & \mathbb{E}_3 & \longrightarrow & \mathbb{E}_2 & \longrightarrow & \mathbb{E}_1 \end{array}$$

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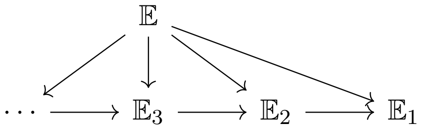

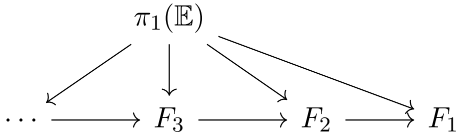

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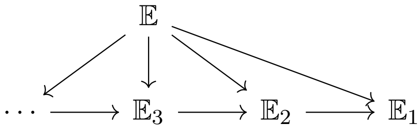

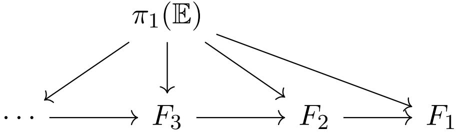

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Theorem (Shape injectivity, Morgan–Morrison [6])

φ is injective.

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An element of $\varprojlim_n F_n$ is a coherent sequence of reduced words $\vec{w} = (\cdots, w_3, w_2, w_1)$, such as:

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Theorem (Morgan–Morrison [6])

$\vec{w}$ is in the image of $\varphi : \pi_1(\mathbb{E}) \to \varprojlim_n F_n$ iff for each k, the number of times that g_k appears in w_n is eventually constant.

Recap

- $\sim_X$, homotopy of loops in X, is an analytic equivalence relation on the loop space ΩX.
- If X is a path-connected and locally path-connected compact metric space, then either $\sim_X$ Borel reduces to id_ω or id_{2^ω} continuously reduces to $\sim_X$.
- $\sim_{\mathbb{E}}$ is smooth. In fact there is a reduction to $\mathrm{id}_{\mathbb{R}}$ with Borel image, so the reduction is *faithful*.

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Question

What can we say about the complexity of $\sim_X$ in general, say for a path-connected Polish space X?

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Both a proof of Theorem 4.1 and a magnificent 3-dimensional picture of K_E will appear in Becker [3].

Becker's Theorem

There is a continuous map $T \mapsto K_T$ from Tr (trees on ω) to $\mathcal{K}([0,1]^2)$ such that $T \in IF \Leftrightarrow K_T$ is path-connected. See Theorems 33.17 and 37.11 of Kechris [5].

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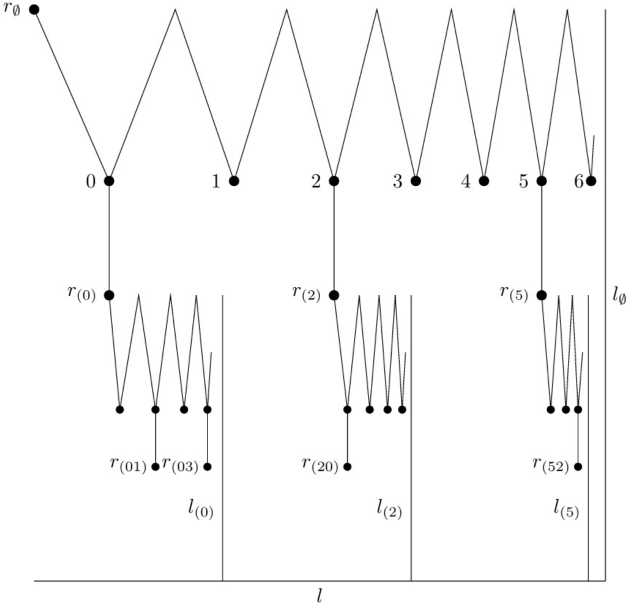

Becker's Theorem

Idea: Let E be an analytic equivalence relation on 2^ω. Code E as $A \subseteq 2^\omega$. There is a tree T on $2 \times \omega$ s.t. $c \in A \Leftrightarrow T_c \in IL$. Consider the map $c \mapsto K_{T_c}$.

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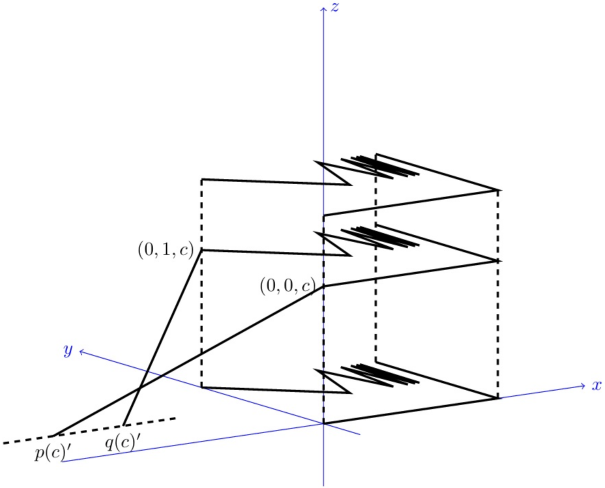

Free group on X/E

Suppose E is an analytic equivalence relation on X. Consider the Polish space $F(X)$ of all reduced words in X. Let N_E be the normal subgroup generated by $\{x^{-1}y : xEy\}$, and $F(E)$ be the coset equivalence relation. $F(X)/F(E)$ is naturally identified with the free group on X/E.

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Theorem (W. [11])

For any analytic equivalence relation E, there is a path-connected compact $L \subseteq \mathbb{R}^5$ s.t. $\sim_L$ is Borel bireducible with $F(E)$.

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In particular, if $E \sim_B F(E)$ then E can be realized as $\sim_L$. Examples include E_0, E_∞, E_1, the universal AER, etc.

Non-examples

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Then f and g induce injections between X/E and $\Omega Y/ \sim_Y = \pi_1(Y)$, so by Cantor–Bernstein there is a bijection between X/E and $\pi_1(Y)$. In particular X/E is a group.

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So if all sets in $L(\mathbb{R})$ are Lebesgue measurable, then $\mathrm{id}_{\mathbb{R}} \sqcup \mathrm{id}_{\omega_1}$ is not bireducible with $\sim_Y$ for any Y, in any reasonable sense: Borel, Δ^1_2, etc.

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If $h_1 \not\sim_Y h_2$ then $g(f(x) \cdot h_1) \not E \; g(f(x) \cdot h_2)$, $\forall x \in X$. So there are many "non-equivalent" self-reductions of E.

Non-examples

Suppose E is an AER on X and $f : (X, E) \to (\Omega Y, \sim_Y)$ and $g : (\Omega Y, \sim_Y) \to (X, E)$ are reductions.

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Thomas [9, 10] showed that there is a CBER E with an ergodic measure, such that for any self reduction f of E, we have $f(x) E x$ on a measure one set. Such an E cannot be bireducible with $\sim_Y$.

Questions

Question 1

Is the dimension 5 *optimal?*

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Question 2

What contraints do topological properties such as local path-connectedness put on the complexity of $\sim_X$*?*

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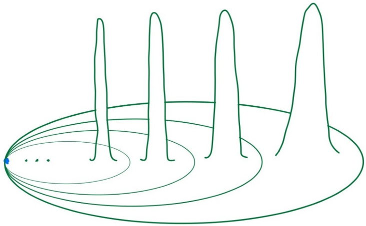
Question 1

Is the dimension 5 optimal?

Question 2

What contraints do topological properties such as local path-connectedness put on the complexity of $\sim_X$?

Consider $\mathbb{HA}$, the harmonic archipelago space:



Questions

It can be shown that $\sim_{\mathbb{HA}}$ is Borel and above E_1.

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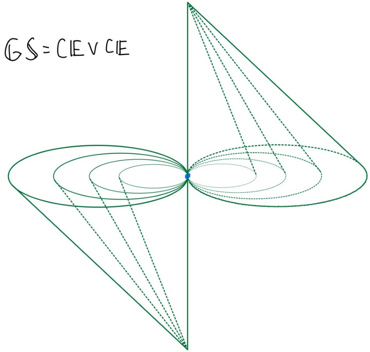
It can be shown that $\sim_{\mathbb{HA}}$ is Borel and above E_1. Also consider $\mathbb{GS}$, the Griffiths twin cone space:



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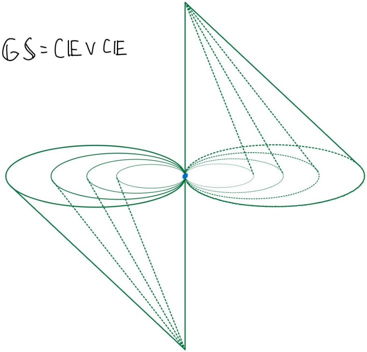

Corson [3] showed in 2025 that $\pi_1(\mathbb{HA})$ and $\pi_1(\mathbb{GS})$ are isomorphic. The proof uses lots of choice, e.g., transfinite induction of length continuum.

Questions

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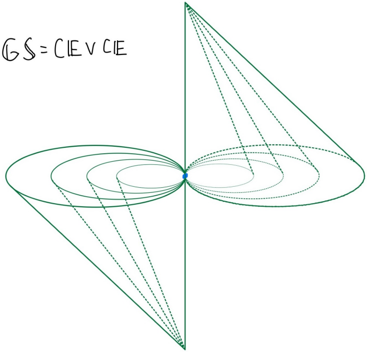

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Question 3

Are $\sim_{\mathbb{HA}}$ and $\sim_{\mathbb{GS}}$ Borel bireducible?

Reference I

Howard Becker.

The number of path-components of a compact subset of $\mathbb{R}^n$.

In *Logic Colloquium'95: Proceedings of the Annual European Summer Meeting of the Association of Symbolic Logic, held in Haifa, Israel, August 9-18, 1995*, volume 11, pages 1–17. Association for Symbolic Logic, 1998.

James W Cannon and Gregory R Conner.

On the fundamental groups of one-dimensional spaces.

Topology and its Applications, 153(14):2648–2672, 2006.

Samuel M Corson.

The double cone group is isomorphic to the archipelago group.

arXiv preprint arXiv:2509.11744, 2025.

J Dydak and Žiga Virk.

An alternate proof that the fundamental group of a peano continuum is finitely presented if the group is countable.

Glasnik matematički, 46(2):505–511, 2011.

Reference II

Alexander Kechris.
Classical descriptive set theory, volume 156.
Springer Science & Business Media, 2012.

John W Morgan and Ian Morrison.
A van kampen theorem for weak joins.
Proceedings of the London Mathematical Society, 3(3):562–576, 1986.

Janusz Pawlikowski.
The fundamental group of a compact metric space.
Proceedings of the American Mathematical Society, 126(10):3083–3087, 1998.

Saharon Shelah.
Can the fundamental (homotopy) group of a space be the rationals?
Proceedings of the American Mathematical Society, 103(2):627–632, 1988.

Reference III

Simon Thomas.
Some applications of superrigidity to borel equivalence.
In *Set Theory: The Hajnal Conference, October 15-17, 1999, DIMACS Center*, volume 58, page 129. American Mathematical Soc., 2002.

Simon Thomas.
Superrigidity and countable borel equivalence relations.
Annals of pure and applied logic, 120(1-3):237–262, 2003.

Fanxin Wu.
Fundamental groups and descriptive set theory.
arXiv preprint arXiv:2512.02355, 2025.